

## THE MINIMAX THEOREM AS HAHN-BANACH THEOREM CONSEQUENCE

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### ABSTRACT

*The separation concept is essential in convex programming. In particular, the convex sets separation. In this work the Hahn-Banach theorem is presented, with great generality, together with an important separation theorem, its consequence. Then are followed the necessary mathematical steps towards the minimax theorem, an important result in the applications of Game Theory to economics, finance and management.*

**Keywords:** *Hahn-Banach theorem, separation theorems, convex programming, minimax theorem*

### INTRODUCTION

The Hahn-Banach theorem is this work key result. It is presented, with great generality, together with an important separation theorem, its consequence. Then these results are rephrased: first for normed spaces and then for a subclass of these spaces: the Hilbert spaces.

The effectiveness of these results is emphasized in the last section where it is shown that they permit to obtain, and fundament mathematically, a very important result in the applications: the minimax theorem, an important result in Game Theory, which consideration in economics, finance and management is becoming trivial.

Other works on this subject are (5 and 11).

### THE HAHN-BANACH-THEOREM

#### Definition 2.1

Consider a vector space  $L$  and its subspace  $L_0$ . Suppose that in  $L_0$  it is defined a linear functional  $f_0$ . A linear functional  $f$  defined in the whole space  $L$  is an extension of the functional  $f_0$  if and only if  $f(x) = f_0(x), \forall x \in L_0$ . ■

**Theorem 2.1 (Hahn-Banach)**

Be  $p$  a positively homogeneous convex functional, defined in a real vector space  $L$ , and  $L_0$  an  $L$  subspace. If  $f_0$  is a linear functional defined in  $L_0$ , fulfilling the condition

$$f_0(x) \leq p(x), \quad \forall x \in L_0 \quad (2.1),$$

there is an extension  $f$  of  $f_0$  defined in  $L$ , linear, and such that  $f(x) \leq p(x), \quad \forall x \in L$ .

**Demonstration:**

Begin showing that if  $L_0 \neq L$ , there is an extension of  $f_0, f'$  defined in a subspace  $L'$ , such that  $L \subset L'$ , in order to fulfill the condition (2.1).

Be  $z \in L - L_0$ ; if  $L'$  is the subspace generated by  $L_0$  and  $z$ , each point of  $L'$  is expressed in the form  $tz + x$ , being  $x \in L_0$ . If  $f'$  is an extension (linear) of the functional  $f_0$  to  $L'$ , it will happen that  $f'(tz + x) = tf'(z) + f_0(x)$  or, making  $f'(z) = c, f'(tz + x) = tc + f_0(x)$ . Now choose  $c$ , fulfilling the condition (2.1) in  $L'$ , that is: in order that the inequality  $f_0(x) + tc \leq p(x + tz)$ , for any  $x \in L_0$  and any real number  $t$ , is accomplished. For  $t > 0$  this inequality is equivalent to the condition  $f_0\left(\frac{x}{t}\right) + c \leq p\left(\frac{x}{t} + z\right)$  or

$$c \leq p\left(\frac{x}{t} + z\right) - f_0\left(\frac{x}{t}\right) \quad (2.2).$$

For  $t < 0$  it is equivalent to the condition  $f_0\left(\frac{x}{t}\right) + c \geq -p\left(-\frac{x}{t} - z\right)$ , or

$$c \geq -p\left(-\frac{x}{t} - z\right) - f_0\left(\frac{x}{t}\right) \quad (2.3).$$

Now it will be proved that there is always  $c$  satisfying simultaneously the conditions (2.2) and (2.3).

Given any  $y'$  and  $y$  belonging to  $L_0$ ,

$$-f_0(y') + p(y' + z) \geq -f_0(y) - p(-y' - z) \quad (2.4),$$

since  $f_0(y') - f_0(y) \leq p(y' - y) = p((y' + z) - (y + z)) \leq p(y' + z) + p(-y - z)$ . Be  $c' = \inf_{y'}(-f_0(y') + p(y' + z))$  and  $c'' = \sup_y(-f_0(y) - p(-y' - z))$ . As  $y'$  and  $y$  are arbitrary, it results from (2.4) that  $c' \geq c''$ . Choosing  $c$  in order that  $c' \geq c \geq c''$ , it is defined the functional  $f'$  on  $L'$  through the formula  $f'(tz + x) = tc + f_0(x)$ . This functional satisfies the condition (2.1). So any functional  $f_0$  defined in a subspace  $L_0 \subset L$  and subject in  $L_0$  to the condition (2.1), may be extended to a subspace  $L'$ . The extension  $f'$  satisfies the condition  $f'(x) \leq p(x), \quad \forall x \in L'$ . If  $L$  has an algebraic numerable base  $(x_1, x_2, \dots, x_n, \dots)$  the functional in  $L$  is built by finite induction, considering the increasing sequence of subspaces  $L^{(1)} = (L_0, x_1), L^{(2)} = (L^{(1)}, x_2), \dots$

designating  $(L^{(k)}, x_{k+1})$  the  $L$  subspace generated by  $L^{(k)}$  and  $x_{k+1}$ . In the general case, that is, when  $L$  has not an algebraic numerable base, it is mandatory to use a transfinite induction process, for instance the Hausdorff maximal chain theorem.

So call  $\mathcal{F}$  the set of the whole pairs  $(L', f')$ , at which  $L'$  is a  $L$  subspace that contains  $L_0$  and  $f'$  is an extension of  $f_0$  to  $L'$  that fulfills (2.1). Order partially  $\mathcal{F}$  so that  $(L', f') \leq (L'', f'')$  if and only if  $L' \subset L''$  and  $f'|_{L'} = f''$ . By the Hausdorff maximal chain theorem, there is a chain, that is: a subset of  $\mathcal{F}$  totally ordered, maximal, that is: not strictly contained in another chain. Call it  $\Omega$ . Be  $\Phi$  the family of the whole  $L'$  such that  $(L', f') \in \Omega$ .  $\Phi$  is totally ordered by the sets inclusion; so, the union  $T$  of the whole elements of  $\Phi$  is a  $L$  subspace. If  $x \in T$  then  $x \in L'$  for some  $L' \in \Phi$ ; define  $\tilde{f}(x) = f'(x)$ , where  $f'$  is the extension of  $f_0$  that is in the pair  $(L', f')$ - the definition of  $\tilde{f}$  is obviously coherent. It is easy to check that  $T = L$  and that  $f = \tilde{f}$  satisfies the condition (2.1). ■

Now it follows the Hahn-Banach theorem complex case, the Hahn contribution to the theorem.

### Theorem 2.2 (Hahn-Banach)

Be  $p$  an homogeneous convex functional defined in a vector space  $L$  and  $f_0$  a linear functional, defined in a subspace  $L_0 \subset L$ , fulfilling the condition  $|f_0(x)| \leq p(x), x \in L_0$ . Then, there is a linear functional  $f$  defined in  $L$ , satisfying the conditions

$$|f(x)| \leq p(x), x \in L; f(x) = f_0(x), x \in L_0.$$

### Demonstration:

Call  $L_R$  and  $L_{0R}$  the real vector spaces underlying, respectively, the spaces  $L$  and  $L_0$ . As it is evident,  $p$  is an homogeneous convex functional in  $L_R$  and  $f_{0R}(x) = \operatorname{Re} f_0(x)$  a real linear functional in  $L_{0R}$  fulfilling the condition  $|f_{0R}(x)| \leq p(x)$  and so,  $f_{0R}(x) \leq p(x)$ . Then, owing to Theorem 2.1, there is a real linear functional  $f_R$ , defined in the whole  $L_R$  space, that satisfies the conditions  $f_R(x) \leq p(x), x \in L_R; f_R(x) = f_{0R}(x), x \in L_{0R}$ . But,  $-f_R(x) = f_R(-x) \leq p(-x) = p(x)$ , and

$$|f_R(x)| \leq p(x), x \in L_R \quad (2.5).$$

Define in  $L$  the functional  $f$  making  $f(x) = f_R(x) - if_R(ix)$ . It is immediate to conclude that  $f$  is a complex linear functional in  $L$  such that  $f(x) = f_0(x), x \in L_0; \operatorname{Re} f(x) = f_R(x), x \in L$ .

It is only missing to show that  $|f(x)| \leq p(x), \forall x \in L$ .

Proceed by absurd: suppose that there is  $x_0 \in L$  such that  $|f(x_0)| > p(x_0)$ . So,  $f(x_0) = \rho e^{i\varphi}, \rho > 0$ , and making  $y_0 = e^{-i\varphi} x_0$ , it would happen that  $f_R(y_0) = \operatorname{Re}[e^{-i\varphi} f(x_0)] = \rho > p(x_0) = p(y_0)$  that is contrary to (2.5). ■

### VECTOR SPACE CONVEX PARTS SEPARATION

The next theorem, a very useful consequence of the Hahn-Banach theorem, is about vector space convex parts separation. Beginning with

#### Definition 3.1

Be  $M$  and  $N$  two subsets of a real vector space  $L$ . A linear functional  $f$  defined in  $L$  separates  $M$  and  $N$  if and only if there is a number  $c$  such that  $f(x) \geq c$ , for  $x \in M$  and  $f(x) \leq c$ , for  $x \in N$  that is, if  $\inf_{x \in M} f(x) \geq \sup_{x \in N} f(x)$ . A functional  $f$  separates strictly the sets  $M$  and  $N$  if and only if  $\inf_{x \in M} f(x) > \sup_{x \in N} f(x)$ . ■

#### Theorem 3.1 (Separation)

Suppose that  $M$  and  $N$  are two convex subsets of a vector space  $L$  such that the kernel of at least one of them, for instance  $M$ , is non-empty and does not intersect the other set; so, there is a linear functional non-null on  $L$  that separates  $M$  and  $N$ .

#### Demonstration:

Less than on translation, it is supposable that the point 0 belongs to the kernel of  $M$ , which is designated  $\dot{M}$ . So, given  $y_0 \in N$ ,  $-y_0$  belongs to the kernel of  $M - N$  and 0 to the kernel of  $M - N + y_0$ . As  $\dot{M} \cap N = \emptyset$ , by hypothesis, 0 does not belong to the kernel of  $M - N$  and  $y_0$  does not belong to the one of  $M - N + y_0$ . Put  $K = M - N + y_0$  and be  $p$  the Minkowsky functional of  $K$ . So  $p(y_0) \geq 1$ , since  $y_0 \notin K$ . Define, now, the linear functional  $f_0(\alpha y_0) = \alpha p(y_0)$ . Note that  $f_0$  is defined in a space with dimension 1, constituted by elements  $\alpha y_0$ , and it is such that  $f_0(\alpha y_0) \leq p(\alpha y_0)$ . In fact,  $p(\alpha y_0) = \alpha p(y_0)$ , when  $\alpha \geq 0$  and  $f_0(\alpha y_0) = \alpha f_0(y_0) < 0 < p(\alpha y_0)$ , when  $\alpha < 0$ . Under these conditions, after the Hahn-Banach theorem, it is possible to state the existence of linear functional  $f$ , defined in  $L$ , that extends  $f_0$ , and such that  $f(y) \leq p(y)$ ,  $\forall y \in L$ . Then it results  $f(y) \leq 1$ ,  $\forall y \in K$  and  $f(y_0) \geq 1$ .

In consequence:

- $f$  separates the sets  $K$  and  $\{y_0\}$ , that is
- $f$  separates the sets  $M-N$  and  $\{y_0\}$ , that is
- $f$  separates the sets  $M$  and  $N$ . ■

### THE HAHN-BANACH THEOREM FOR NORMED SPACES

#### Definition 4.1

Consider a continuous linear functional  $f$  in a normed space  $E$ . It is called  $f$  norm, and designated  $\|f\|$ :  $\|f\| = \sup_{\|x\| \leq 1} |f(x)|$  that is: the supreme of the values assumed by  $|f(x)|$  in the  $E$  unitary ball. ■

**Observation:**

-The class of the continuous linear functionals, with the norm above defined, is a normed vector space, called the  $E$  dual space, designated  $E'$ .

The Theorem 2.1 in normed spaces is:

**Theorem 4.1** (Hahn-Banach)

Call  $L$  a subspace of a real normed space  $E$ . And  $f_0$  a bounded linear functional in  $L$ . So, there is a linear functional defined in  $E$ , extension of  $f_0$ , such that  $\|f_0\|_{L'} = \|f\|_{E'}$ .

**Demonstration:**

It is enough to think in the functional  $K$  satisfying  $K\|x\| = \|f_0\|_{L'}$ . As it is convex and positively homogeneous, it is possible to put  $p(x) = K\|x\|$  and to apply Theorem 2.1. ■

**Observation:**

-To see an interesting geometric interpretation of this theorem, consider the equation  $\|f_0(x)\| = 1$ . It defines, in  $L$ , an hiperplane at distance  $\frac{1}{\|f_0\|}$  of 0. Considering the  $f_0$  extension  $f$ , with norm conservation, it is obtained an hiperplane in  $E$ , that contains the hiperplane considered behind in  $L$ , at the same distance from the origin.

The Theorem 2.2 in normed spaces is:

**Theorem 4.2** (Hahn-Banach)

Be  $E$  a complex normed space and  $f_0$  a bounded linear functional defined in a subspace  $L \subset E$ . So, there is a bounded linear functional  $f$ , defined in  $E$ , such that  $f(x) = f_0(x), x \in L; \|f\|_{E'} = \|f_0\|_{L'}$ . ■

Two separation theorems, important consequences of the Hahn-Banach theorem, applied to the normed vector spaces, are then presented:

**Theorem 4.3** (Separation)

Consider two convex sets  $A$  and  $B$  in a normed space  $E$ . If one of them, for instance  $A$ , has at least on interior point and  $(\text{int}A) \cap B = \emptyset$ , there is a continuous linear functional non-null that separates the sets  $A$  and  $B$ . ■

**Theorem 4.4** (Separation)

Consider a closed convex set  $A$ , in a normed space  $E$ , and a point  $x_0 \in E$ , not belonging to  $A$ . So, there is a continuous linear functional, non-null, that separates strictly  $\{x_0\}$  and  $A$ . ■

**SEPARATION THEOREMS IN HILBERT SPACES**

In a Hilbert space  $H$ ,

**Theorem 5.1** (Riesz representation)

Every continuous linear functional  $f(\cdot)$  may be represented in the form  $f(x) = [x, \tilde{q}]$  where  $\tilde{q} = \frac{\overline{f(q)}}{[q,q]} q$ . ■

From now on, only real Hilbert spaces will be considered.

Note that the separation theorems, seen in the former section, are effective in Hilbert spaces. But, due to the Riesz representation theorem, they may be formulated in the following way:

**Theorem 5.2** (Separation)

Consider two convex sets  $A$  and  $B$  in a Hilbert space  $H$ . If one of them, for instance  $A$ , has at least one interior point and  $(\text{int}A) \cap B = \emptyset$ , there is a non-null vector  $v$  such that  $\sup_{x \in A} [v, x] \leq \inf_{y \in B} [v, y]$ . ■

**Theorem 5.3** (Separation)

Consider a closed convex set  $A$ , in a Hilbert space  $H$ , and a point  $x_0 \in H$ , not belonging to  $A$ . So, there is a non-null vector  $v$ , such that  $[v, x_0] < \inf_{x \in A} [v, x]$ . ■

Another separation theorem:

**Theorem 5.4** (Separation)

Two closed convex subsets  $A$  and  $B$ , in a Hilbert space, at finite distance, that is: such that:  $\inf_{x \in A, y \in B} \|x - y\| = d > 0$  may be strictly separated:  $\inf_{x \in A} [v, x] > \sup_{y \in B} [v, y]$ . ■

It is also possible to establish that:

**Theorem 5.5** (Separation)

Being  $H$  a finite dimension Hilbert space, if  $A$  and  $B$  are disjoint and non-empty convex sets they always may be separated. ■

## MINIMAX THEOREM

In a two players game with null sum be  $\Phi(x, y)$  a real function of two variables  $x, y \in H$  and  $A$  and  $B$  convex sets in  $H$ . One of the players chooses strategies (points) in  $A$  in order to maximize  $\Phi(x, y)$  (or minimize  $-\Phi(x, y)$ ): it is the maximizing player. The other player chooses strategies (points) in  $B$  in order to minimize  $\Phi(x, y)$  (or maximize  $-\Phi(x, y)$ ): it is the minimizing player. The function  $\Phi(x, y)$  is the payoff function.  $\Phi(x_0, y_0)$  represents, simultaneously, the gain of the maximizing player and the loss of the minimizing player in a move at which they chose, respectively the strategies  $x_0$  and  $y_0$ . So, the gain of one of the players is equal to the other's loss. That is why the game is a null sum game. A game in these conditions value is  $c$  if

$$\sup_{x \in A} \inf_{y \in B} \Phi(x, y) = c = \inf_{y \in B} \sup_{x \in A} \Phi(x, y) \quad (6.1).$$

If, for any  $(x_0, y_0)$ ,  $\Phi(x_0, y_0) = c$ ,  $(x_0, y_0)$  is a pair of optimal strategies. There will be a saddle point if also

$$\Phi(x, y_0) \leq \Phi(x_0, y_0) \leq \Phi(x_0, y), x \in A, y \in B \quad (6.2).$$

**Theorem 6.1**

Consider  $A$  and  $B$  closed convex sets in  $H$ , being  $A$  bounded. Be  $\Phi(x, y)$  a real functional defined for  $x$  in  $A$  and  $y$  in  $B$  fulfilling:

-  $\Phi(x, (1 - \theta)y_1 + \theta y_2) \leq (1 - \theta)\Phi(x, y_1) + \theta\Phi(x, y_2)$  for  $x$  in  $A$  and  $y_1, y_2$  in  $B$ ,  $0 \leq \theta \leq 1$  (that is:  $\Phi(x, y)$  is convex in  $y$  for each  $x$ ),

-  $\Phi((1 - \theta)x_1 + \theta x_2, y) \geq (1 - \theta)\Phi(x_1, y) + \theta\Phi(x_2, y)$  for  $y$  in  $B$  and  $x_1, x_2$  in  $A$ ,  $0 \leq \theta \leq 1$  (that is:  $\Phi(x, y)$  is concave in  $x$  for each  $y$ ),

-  $\Phi(x, y)$  is continuous in  $x$  for each  $y$ ,

so (7.1) holds, that is: the game has a value. ■

The next corollary follows from the Theorem 6.1 hypothesis strengthen:

**Corollary 6.1 (Minimax)**

Suppose that the functional  $\Phi(x, y)$  of Theorem 6.1 is continuous in both variables, separately, and that  $B$  is also bounded. Then, there is an optimal pair of strategies, with the property of being a saddle point. ■

**CONCLUSIONS**

The Hahn-Banach theorem was presented with great generality, real and complex version, followed by an important separation theorem, consequence of it.

These results were specified for normed spaces and then for a subclass of these spaces: the Hilbert spaces. Better saying, they were reformulated for Hilbert spaces using the Riesz representation theorem.

Examples of the fruitfulness of the results presented are patent in the last section, where it is shown that they permit to obtain an important result: the minimax theorem, which consideration in economic, finance and management problems resolution is greater and greater. Now the structures considered were the real Hilbert spaces.

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